Instantaneous mean-variance hedging and instantaneous Sharpe ratio pricing in a regime-switching financial model

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Motivation

- Pricing and hedging in incomplete markets,
- Development of practically relevant and mathematically tractable approaches to pricing and hedging,
- Empirical studies show that regime-switching models can explain empirical behaviors of many economic and financial data,
- The use of regime-switching models has been recommended by the American Academy of Actuaries and the Canadian Institute of Actuaries.

The probability space

- We deal with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ and a finite time horizon $T < \infty$,
- We consider an $\mathcal{F}$-adapted Brownian motion $W = (W(t), 0 \leq t \leq T)$ and an $\mathcal{F}$-adapted multivariate counting process $N = (N_1(t), ..., N_I(t), 0 \leq t \leq T)$,
- We assume that the weak property of predictable representation holds, i.e. every $(\mathbb{P}, \mathcal{F})$ local martingale M has the representation

$$M(t) = M(0) + \int_0^t \mathcal{Z}(s) dW(s) + \int_0^t \sum_{i=1}^{I} \mathcal{U}_i(s) d\tilde{N}_i(s) \quad 0 \leq t \leq T,$$

with $\mathcal{F}$-predictable processes $(\mathcal{Z}, \mathcal{U}_1, ..., \mathcal{U}_I)$

The switching behavior

- We consider an economy which can be in one of I states (regimes),
- For $i = 1, ..., I$, the counting process N_i counts the number of transitions of the economy into the state i,
- We assume that the counting process N_i has an $\mathcal{F}$-predictable intensity $\lambda_i(t)$,
- We define the compensated counting process

$$\tilde{N}_i(t) = N_i(t) - \int_0^t \lambda_i(s)ds, \quad 0 \leq t \leq T, \; i = 1, ..., I.$$

The financial market

- The bank account:

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \quad S_0(0) = 1,$$

- The risky asset:

$$\frac{dS(t)}{S(t-)} = \mu(t)dt + \sigma(t)dW(t) + \sum_{i=1}^{I} \gamma_i(t)d\tilde{N}_i(t), \quad S(0) = 1,$$

- The coefficients $r, \mu, \sigma, (\gamma_i)_{i=1,\dots,I}$ are $\mathcal{F}$-predictable, bounded processes.

The financial market

A Markov-regime-switching model:

$$\begin{aligned}\frac{dS_0(t)}{S_0(t)} &= r(J(t-))dt,\\ \frac{dS(t)}{S(t-)} &= \mu(J(t-))dt + \sigma(J(t-))dW(t) + \sum_{i=1}^{I}\gamma_i(J(t-))d\tilde{N}_i(t),\end{aligned}$$

where the counting process N_i has intensity $\lambda_i(t) = \lambda_i(J(t-), S(t-))$

and J indicates the current state of the economy.

The financial market

- We define

$$|\delta(t)|^2 = |\sigma(t)|^2 + \sum_{i=1}^{I} |\gamma_i(t)|^2 \lambda_i(t),$$
$$\theta(t) = \frac{\mu(t) - r(t)}{\delta(t)}.$$

- We assume

$$\mu(t) \geq r(t), \quad 0 \leq t \leq T,$$
$$|\delta(t)|^2 = |\sigma(t)|^2 + \sum_{i=1}^{I} |\gamma_i(t)|^2 \lambda_i(t) \geq \epsilon > 0, \quad 0 \leq t \leq T,$$
$$\gamma_i(t) > -1, \quad 0 \leq t \leq T, \ i = 1, ..., I.$$

The claim

- Let ξ be a contingent claim in the regime-switching financial market which has to be covered at time T,
- We assume that the price process $Y := (Y(t), 0 \leq t \leq T)$ of the claim ξ solves the BSDE:

$$\begin{aligned} Y(t) &= \xi + \int_t^T \big(- Y(s-)r(s) - f(s)\big) ds \\ &\quad - \int_t^T Z(s)dW(s) - \int_t^T \sum_{i=1}^I U_i(s) d\tilde{N}_i(s), \quad 0 \leq t \leq T. \end{aligned}$$

The hedging strategy

- Let $\pi = (\pi(t), 0 \leq t \leq T)$ denote an admissible hedging strategy, i.e. a square integrable, $\mathcal{F}$-predictable process,
- The dynamics of the hedging portfolio $X^\pi := (X^\pi(t), 0 \leq t \leq T)$ under an admissible π is given by the stochastic differential equation

$$\begin{aligned} dX^\pi(t) &= \pi(t)\big(\mu(t)dt + \sigma(t)dW(t) + \sum_{i=1}^{I} \gamma_i(t)d\tilde{N}_i(t)\big) \\ &\quad + (X^\pi(t-) - \pi(t))r(t)dt, \\ X^\pi(0) &= x. \end{aligned}$$

Instantaneous mean-variance hedging and instantaneous Sharpe ratio pricing

- We define the surplus process $S^{\pi}(t) = X^{\pi}(t) - Y(t),\ 0 \le t \le T$.

  The dynamics of the surplus S^{π} is given by the stochastic differential equation

$$
\begin{aligned}
dS^{\pi}(t) &= \big(\pi(t)(\mu(t) - r(t)) + S^{\pi}(t-)r(t) - f(t)\big)dt \\
&\quad + (\pi(t)\sigma(t) - Z(t))dW(t) \\
&\quad + \sum_{i=1}^{I}(\pi(t)\gamma_i(t) - U_i(t))d\tilde{N}_i(t).
\end{aligned}
$$

Instantaneous mean-variance hedging and instantaneous Sharpe ratio pricing

- Our goal is to find an admissible hedging strategy π which minimizes the instantaneous mean-variance risk of the surplus

$$\rho(S^{\pi}) = L(t)\sqrt{\mathbb{E}\big[d[S^{\pi},S^{\pi}](t)|\mathcal{F}_{t-}\big]/dt}$$
$$-\big(\mathbb{E}\big[dS^{\pi}(t)-S^{\pi}(t-)r(t)dt|\mathcal{F}_{t-}\big]/dt\big),$$

for all $t\in(0,T]$,

Instantaneous mean-variance hedging and instantaneous Sharpe ratio pricing

- Set the price of ξ (find the generator f of the BSDE) in such a way that the instantaneous Sharpe ratio of the surplus equals a predefined target L, i.e.

$$\frac{\mathbb{E}\big[dS^{\pi}(t) - S^{\pi}(t-)r(t)dt|\mathcal{F}_{t-}\big]/dt}{\sqrt{\mathbb{E}\big[d[S^{\pi}, S^{\pi}](t)|\mathcal{F}_{t-}\big]/dt}} = L(t),$$

for all $t \in (0, T]$.

No-good-deal pricing

- The no-good-deal price of the claim ξ is defined as a solution to the optimization problem:

$$Y(t) = \sup_{(\psi,\phi)\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}^{\psi,\phi}}\left[e^{-\int_t^T r(s)ds}\xi|\mathcal{F}_t\right], \quad 0 \le t \le T,$$

where

$$\begin{aligned}
&\frac{d\mathbb{Q}^{\psi,\phi}}{d\mathbb{P}}\Big|\mathcal{F}_t \\
&= \quad \mathcal{E}\Big(-\int_0^t \psi(s)dW(s) - \sum_{i=1}^{I}\int_0^t \phi_i(s)d\tilde{N}_i(s)\Big), \\
&\qquad 0 \le t \le T, \quad (\psi,\phi)\in\mathcal{Q},
\end{aligned}$$

No-good-deal pricing

$$\begin{aligned}\mathcal{Q} &= \Big\{\mathcal{F}-\textit{predictable processes } (\psi,\phi)=(\psi,\phi_1,...,\phi_I) \textit{ such that}\\ &|\psi(t)|^2+\sum_{i=1}^{I}|\phi_i(t)|^2\lambda_i(t)\leq|L(t)|^2,\\ &\psi(t)\sigma(t)+\sum_{i=1}^{I}\phi_i(t)\gamma_i(t)\lambda_i(t)=\mu(t)-r(t),\\ &\phi_i(t)<1,\quad 0\leq t\leq T,\ i=1,...,I\Big\}.\end{aligned}$$

Robust pricing and hedging under model ambiguity

- The price and the hedging strategy are defined as a solution to the robust optimization problem:

$$Y(t) = \inf_{\pi\in\mathcal{A}} \sup_{(\psi,\phi)\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}^{\psi,\phi}}\Big[-\big(e^{-\int_t^T r(s)ds}X^{\pi}(T)-X(t) \\ -e^{-\int_t^T r(s)ds}F\big)\Big|\mathcal{F}_t\Big], \quad 0\leq t\leq T,$$

where

$$\frac{d\mathbb{Q}^{\psi,\phi}}{d\mathbb{P}}\Big|\mathcal{F}_t \\ = \mathcal{E}\Big(-\int_0^t\psi(s)dW(s)-\sum_{i=1}^{I}\int_0^t\phi_i(s)d\tilde{N}_i(s)\Big), \\ 0\leq t\leq T, \quad (\psi,\phi)\in\mathcal{P},$$

Robust pricing and hedging under model ambiguity

$$\mathcal{P} = \Big\{\mathcal{F} - \textit{predictable processes } (\psi,\phi) = (\psi,\phi_1,...,\phi_I) \textit{ such that}$$
$$|\psi(t)|^2 + \sum_{i=1}^{I}|\phi_i(t)|^2\lambda_i(t) \leq |L(t)|^2,$$
$$\phi_i(t) < 1, \quad 0 \leq t \leq T,\ i = 1,...,I\Big\}.$$

The solution

Theorem

Let ξ be an $\mathcal{F}$-measurable claim such that $\mathbb{E}[|\xi|^2] < \infty$, and let L be an $\mathcal{F}$-predictable process such that $L(t) \geq \theta(t) + \epsilon,\ 0 \leq t \leq T,\ \epsilon > 0$.

Consider the BSDE:

$$Y(t) = \xi + \int_t^T \Big(- Y(s)r(s) - \frac{Z(s)\sigma(s) + \sum_{i=1}^I U_i(s)\gamma_i(s)\lambda_i(s)}{\delta(s)}\theta(s)$$
$$+ \sqrt{|L(s)|^2 - |\theta(s)|^2}$$
$$\cdot \sqrt{|Z(s)|^2 + \sum_{i=1}^I |U_i(s)|^2\lambda_i(s) - \frac{|Z(s)\sigma(s) + \sum_{i=1}^I U_i(s)\gamma_i(s)\lambda_i(s)|^2}{|\delta(s)|^2}} \Big)ds$$
$$- \int_t^T Z(s)dW(s) - \int_t^T \sum_{i=1}^I U_i(s)d\tilde{N}_i(s), \quad 0 \leq t \leq T.$$

The solution

Theorem

The optimal admissible hedging strategy π^* for ξ is of the form

$$\pi^*(t) = \frac{Z(t)\sigma(t) + \sum_{i=1}^{I} U_i(t)\gamma_i(t)\lambda_i(t)}{|\delta(t)|^2}$$
$$+ \frac{\theta(t)}{\delta(t)\sqrt{|L(t)|^2 - |\theta(t)|^2}}$$
$$\cdot \sqrt{|Z(t)|^2 + \sum_{i=1}^{I} |U_i(t)|^2\lambda_i(t) - \frac{|Z(t)\sigma(t) + \sum_{i=1}^{I} U_i(t)\gamma_i(t)\lambda_i(t)|^2}{|\delta(t)|^2}},$$

and the price process of ξ is given by Y.

The arbitrage-free representation of the price

Theorem

Let

$$\lambda_i(t) > 0 \Longrightarrow U_i(t) \neq 0, \quad 0 \leq t \leq T,\ i = 1, ..., I,$$

and

$$\sqrt{|L(t)|^2 - |\theta(t)|^2} + \frac{|\gamma_i(t)|\sqrt{\lambda_i(t)}}{\delta(t)}\theta(t) < \sqrt{\lambda_i(t)}, \quad 0 \leq t \leq T,\ i = 1, ..., I,$$

on the set $\{\lambda_i(t) > 0\}$.

The arbitrage-free representation of the price

Theorem

The optimal equivalent martingale measure $\mathbb{Q}^{\psi^*,\phi^*}$ which solves the no-good-deal pricing problem is determined by the processes

$$\begin{aligned}\psi^*(t) &= \theta(t)\mathbf{1}\{\forall_{i=1,\dots,I}\lambda_i(t)=0\} \\ &\quad + \frac{Z(t)-\sigma(t)K_1^*(t,Z(t),U(t))}{2K_2^*(t,Z(t),U(t))}\mathbf{1}\{\exists_{i=1,\dots,I}\lambda_i(t)>0\}, \quad 0\le t\le T, \\ \phi_i^*(t) &= \frac{U_i(t)-\gamma_i(t)K_1^*(t,Z(t),U(t))}{2K_2^*(t,Z(t),U(t))}\mathbf{1}\{\lambda_i(t)>0\}, \quad 0\le t\le T, i=1,\dots,\end{aligned}$$

The arbitrage-free representation of the price

Theorem

$$K_2^*(t, Z(t), U(t))$$

$$= -\frac{1}{2}\frac{\sqrt{|Z(t)|^2 + \sum_{i=1}^{I}|U_i(t)|^2\lambda_i(t) - \frac{|Z(t)\sigma(t)+\sum_{i=1}^{I}U_i(t)\gamma_i(t)\lambda_i(t)|^2}{|\delta(t)|^2}}}{\sqrt{|L(t)|^2 - |\theta(t)|^2}},$$

$$K_1^*(t, Z(t), U(t))$$

$$= -\frac{\theta(t)}{\delta(t)}2K_2^*(t, Z(t), U(t)) + \frac{Z(t)\sigma(t) + \sum_{i=1}^{I}U_i(t)\gamma_i(t)\lambda_i(t)}{|\delta(t)|^2}.$$

Moreover, we have

$$Y(t) = \sup_{(\psi,\phi)\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}^{\psi,\phi}}\left[e^{-\int_t^T r(s)ds}\xi|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^{\psi^*,\phi^*}}\left[e^{-\int_t^T r(s)ds}\xi|\mathcal{F}_t\right].$$

A comparison principle

Theorem

Assume that

$$\frac{|\sigma(t)|^2+\sum_{j\neq i}|\gamma_j(t)|^2\lambda_j(t)}{|\delta(t)|^2}\big(|L(t)|^2-|\theta(t)|^2\big)<\lambda_i(t),\quad 0\leq t\leq T,\ i=1,...,I$$

on the set $\{\gamma_i(t)=0,\ \lambda_i(t)>0\}$, *and*

$$\frac{|\sigma(t)|^2+\sum_{j\neq i}|\gamma_j(t)|^2\lambda_j(t)}{|\delta(t)|^2}\big(|L(t)|^2-|\theta(t)|^2\big)+\frac{|\gamma_i(t)|^2\lambda_i(t)}{|\delta(t)|^2}|\theta(t)|^2<\frac{\lambda_i(t)}{2}\quad 0\leq t\leq T,\ i=1,...,I,$$

on the set $\{\gamma_i(t)\neq 0,\ \lambda_i(t)>0\}$.

If $\xi\leq\xi'$ *and* $L(t)\leq L'(t),\ 0\leq t\leq T$, *then* $Y(t)\leq Y'(t),\ 0\leq t\leq T$.

Insurance application

- Let τ denote the future lifetime of a policyholder. The random variable τ is exponentially distributed with mortality intensity m. Let us set $M(t) = \mathbf{1}\{\tau \geq t\}$ and $\tilde{M}(t) = M(t) - \int_0^t (1 - M(s-))m(s)ds, \ 0 \leq t \leq T$,
- Let ξ be an equity-linked insurance claim (a claim contingent on the financial risk and the insurance risk).

Insurance application

The price process of ξ is given by the solution to the BSDE:

$$Y(t) = \xi + \int_t^T \Big(- Y(s)r(s) - \frac{Z(s)\sigma(s) + \sum_{i=1}^{I} U_i(s)\gamma_i(s)\lambda_i(s)}{\delta(s)}\theta(s)$$

$$+\sqrt{|L(s)|^2 - |\theta(s)|^2}$$

$$\cdot\sqrt{|Z(s)|^2 + \sum_{i=1}^{I} |U_i(s)|^2\lambda_i(s) - \frac{|Z(s)\sigma(s) + \sum_{i=1}^{I} U_i(s)\gamma_i(s)\lambda_i(s)|^2}{|\delta(s)|^2} + |U_m(s)|^2 m(s)}\ \Big)$$

$$-\int_t^T Z(s)dW(s) - \int_t^T \sum_{i=1}^{I} U_i(s)d\tilde{N}_i(s) - \int_t^T U_m(s)d\tilde{M}(s), \quad 0 \le t \le T.$$

Insurance application

The optimal admissible hedging strategy π^* for ξ is of the form:

$$\pi^*(t) = \frac{Z(t)\sigma(t) + \sum_{i=1}^{I} U_i(t)\gamma_i(t)\lambda_i(t)}{|\delta(t)|^2}$$

$$+ \frac{\theta(t)}{\delta(t)\sqrt{|L(t)|^2 - |\theta(t)|^2}}$$

$$\cdot \sqrt{|Z(t)|^2 + \sum_{i=1}^{I} |U_i(t)|^2\lambda_i(t) - \frac{|Z(t)\sigma(t) + \sum_{i=1}^{I} U_i(t)\gamma_i(t)\lambda_i(t)|^2}{|\delta(t)|^2} + |U_m(t)|^2 m(}$$

Numerical example

Table: The parameters of the Markov-regime-switching model.

State i	$r(i)$	$\mu(i)$	$\sigma(i)$	$\gamma_{.}(i)$	$\lambda_{.}(i)$
1	0.03	0.07	0.1	-0.1	2
2	0.01	0,02	0.25	0.05	5

Numerical example

Table: The prices of the call options with strike Q and the Sharpe ratios $L(1) = 0.4, L(2) = 0.2$ in the Markov-regime-switching model.

The strike	$Q = 80$	$Q = 90$	$Q = 100$	$Q = 110$	$Q = 120$
The price	24.561	16.677	10.381	5.857	2.928

Numerical example

Table: The prices of the call options with strike Q in the complete Black-Scholes model with parameters (r, σ).

The strike	$Q = 80$	$Q = 90$	$Q = 100$	$Q = 110$	$Q = 120$
The price for $(0.03, 0.1)$	22.381	13.038	5.581	1.596	0.299
The price for $(0.01, 0.25)$	22.891	15.830	10.405	6.532	3.948

Thank you very much.